Math 10B with Professor Stankova
Worksheet, Discussion #17; Thursday, 3/15/2018
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1 Recursion Equations

1.1 Concepts

1. A **homogeneous** recursion does not include any extra constants (e.g. $a_n = a_{n-1} + a_{n-2}$) and a **nonhomogeneous** recursion contains one (e.g. $a_n = a_{n-1} + 4$). The **order** of a recursion equation is the "farthest" back the relation goes. For instance, the order of $a_n = a_{n-1} + a_{n-3}$ is 3 because we need the term 3 terms back (a_{n-3}).

   The general solution of a first order equation $a_n = a_{n-1} + d$ is $a_n = a_0 + nd$.

   In order to solve a linear homogeneous we can replace the equation with its characteristic polynomial. For instance, the characteristic polynomial of $a_n = 2a_{n-1} + a_{n-2}$ is $\lambda^2 = 2\lambda + 1$. Then if $\lambda_1, \ldots, \lambda_k$ are roots of this polynomial, then the general form of the solution is $a_n = C_1\lambda_1^n + \cdots + C_k\lambda_k^n$.

   The Δ operator takes in a series and spits out a new one. By definition, we have that $\Delta a_n = a_{n+1} - a_n$. This is done to change linear non-homogeneous equations into homogeneous ones.

1.2 Examples

2. Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ with $a_1 = 5, a_2 = 13$.

3. Solve the recurrence relation $a_n = 2a_{n-1} + 1$ with $a_0 = 0$.

1.3 Problems

4. True False We are not given an easy formula to plug in to solve linear non-homogeneous recursion equations.

5. True False If a_n, b_n are two solutions to a linear homogeneous equation, then $a_n + b_n$ is also an equation.

6. True False If a_n is a solution to a linear homogeneous equation, then ca_n is also a solution for any constant c.

7. True False If a_n, b_n are two solutions to a linear non-homogeneous equation, then $a_n + b_n$ is also an equation.

8. True False If a_n is a solution to a linear non-homogeneous equation, then ca_n is also a solution for any constant c.

9. Verify that $a_n = \binom{n}{5}$ is a solution to $a_n = \frac{n}{n-5}a_{n-1}$.
10. Solve the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$ with $a_0 = 3$ and $a_1 = 2$.
11. Find A, B such that $a_n = An + B$ is a solution to the recurrence relation $2a_n = a_{n-1} + 2a_{n-2} + n$.

1.4 Extra Problems

12. Verify that $a_n = n^2$ is a solution to $a_n = a_{n-1} + 2n - 1$.
13. Solve the recurrence relation $a_n = 4a_{n-1} + 5a_{n-2}$ with $a_0 = 3$ and $a_1 = 3$.
14. Find A, B such that $a_n = An + B$ is a solution to the recurrence relation $3a_n = a_{n-1} + 3a_{n-2} + n + 5$.

2 Differential Equations

2.1 Concepts

15. The **order** of a differential equation is the highest derivative that appears in the equation. For instance, the equation $y''' + \sqrt{y'} = t^2 y$ is third order.

    A problem of the form $y' = f(t, y)$ and $y(0) = y_0$ is called an **initial value problem (IVP)**. There is a theorem that tells us when a solution to this problem exists. It says that if f is continuous, then for every choice of y_0, the solution **exists** in a time interval $[0, T)$ for some $0 < T \leq \infty$. But, the solution may not exist everywhere and it is not guaranteed to be unique.

    If in addition f satisfies the **Lipschitz** condition (that $|f(t, y) - f(t, z)| \leq C|y - z|$) for some constant C and all y, z), then the solution is **unique** and exists for all $t \geq 0$. For instance $f(y) = y^2$ does not satisfy the Lipschitz condition because there is no constant such that $|y^2 - 0^2| \leq C|y - 0| = C|y|$ for all y. Effectively this is saying that f does not grow or shrink faster than a linear function.

2.2 Examples

16. Bacteria grows at a rate $N' = 0.05N$ where time is measured in hours. If initially there were 1000 cells, how many cells will there be in 10 hours?

2.3 Problems

17. True False For an IVP, the function f may be continuous everywhere but still the solution does not exist everywhere.

18. True False We guaranteed that the IVP $y' = ty^2$, $y(0) = 0$ has a unique solution.

19. True False We guaranteed that the IVP $y' = t^2y$, $y(0) = 0$ has a unique solution.

20. Solve the IVP $y' = te^t$ with $y(0) = 0$.

21. Verify that $y = te^t + 1$ is a solution to $y'' - 2y' = 1 - y$.

2.4 Extra Problems

22. Solve the IVP $y' = \frac{1}{t\ln t}$ with $y(e) = 0$.

23. Verify that $y = 2e^{1/(2t)}$ is a solution to $2t^2y' + y = 0$.

3 True/False Review

24. True False To find $p(B|A)$, it suffices to know just $p(A|B)$ and how to apply Bayes' Theorem.

25. True False Among other things, the proof of Bayes' Theorem for finding $p(B|A)$ depends on being able to split the probability $p(A)$ as a sum probabilities $p(A \cap B)$ and $p(A \cap \overline{B})$, and then further rewrite these as products of certain other probabilities.

26. True False The extra shortcut formula $p(B|A) = \frac{1}{1+\frac{p(A|\overline{B})\cdot p(\overline{B})}{p(A|B)\cdot p(B)}}$ works in one particular case when the standard formula for $p(B|A)$ in Bayes' Theorem fails.

27. True False If a winner in a bicycle race tests positive for steroids, and this test has a very high "True Positive" rate and hence a very low "False positive" rate, then we should take away the winning cup from the athlete because it is extremely likely that he/she has used steroids.

28. True False Error 1 in Hypothesis Testing (reject the null-hypothesis that the person is healthy when the person is actually healthy) is analogous to Testing positive for steroids (event T), yet not having used steroids (event $\overline{S}$); in other words, the significance α corresponds to $p(T \cap \overline{S})$.

29. True False Error 2 in Hypothesis Testing (keep the null-hypothesis that the person is healthy but the person is, in fact, sick) is analogous to Testing negative for steroids (event $\overline{T}$), yet having used steroids (event S); in other words, the power of a test $1 - \beta$ corresponds to $1 - p(\overline{T} \cap S)$.

30. True False To partition a set Ω into a disjoint union of subsets $B_1, B_2, \ldots, B_n$, means that the intersection of these sets is empty; i.e., $B_1 \cap B_2 \cap \cdots \cap B_n = \emptyset$.

31. True False Two disjoint events could be independent, but two independent events can never be disjoint.

32. True False If a fair coin comes up Heads six times in a row, it is more likely that it will come up Tails than Heads on the 7th flip.

33. True False Contrary to how we may use the word "dependent" in everyday life; e.g., event A could be dependent on event B, yet event B may not be dependent on event A; in math "dependent" is a symmetric relation; i.e., A is dependent with B if and only B is dependent with A.

34. True False If A and B are independent events, their complements are also bound to be independent, and to prove this we need a general argument since an example is not sufficient here.

35. True False If A and B are independent events, $\overline{A}$ and B may fail to be independent, but to prove this we need just one counterexample, not a general proof.

36. True False If any pair of events among $A_1, A_2, ..., A_n$ are independent, then all events are independent.

37. True False A random variable (RV) on a probability space (Ω, P) is a function $X : \Omega \to \mathbb{R}$ that satisfies certain rules and is related to the probability function P.

38. True False A RV X could be the only source of data for an outcome space Ω and hence could be very useful in understanding better X 's domain.